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Integrable top equations associated with projective geometry over Z_2

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Abstract. We give a series of integrable top equations associated with the projective geometry over Z_2 as a $(2^n - 1)$-dimensional generalization of the three-dimensional Euler top equations. The general solution of the $(2^n - 1)$-dimensional top is shown to be given by an integration over a Riemann surface with genus $(2^{n-1} - 1)^2$.

1. $(2^n - 1)$-dimensional top equations

Recently we discovered an apparently new integrable set of evolution equations in seven dimensions, which are an analogue of the well known three-dimensional Euler top [1, 2]. The seven-dimensional top arises from the dimensional reduction of the eight-dimensional $spin(7)$ invariant self-dual Yang–Mills (SDYM) equations in [3], just as the three-dimensional top comes from the reduction of the four-dimensional SDYM to differential equations depending only upon one variable. The integrability of the three-dimensional top is ensured by the existence of the Lax formulation of the four-dimensional SDYM [4], while there is no such first-order structure behind the eight-dimensional SDYM [5]. Nevertheless the seven-dimensional top has been shown to have sufficient conserved quantities to permit full integrability [1] and its general solution is given by a non-hyperelliptic differential equation corresponding to a Riemann surface with genus 9 [2].

The derivation of the top equations from the SDYM shows their connection with the existence of the division algebras, the three-dimensional system arising from the quaternionic algebra, the seven-dimensional one from the octonions, which seems to suggest that no further integrable top system in more than seven dimensions should exist. In this paper, however, we demonstrate that a generalization of our previous results to general $2^n - 1$ dimensions is possible and is associated rather with the n-dimensional projective space over Z_2.

We take the projective space Z_2P_{n-1} with homogeneous coordinates $(z_0, z_1, \ldots, z_{n-1})$, where z_i is either 0, 1 and calculations are performed in arithmetic mod 2. The space Z_2P_{n-1} consists of a finite number of points e_i $(i = 1, \ldots, 2^n - 1)$ with the multiplication operation e_ie_j defined by the sum of their associated coordinates.

For the three-dimensional $(n = 2)$ case, we have three points,

$$e_1 = (0, 1) \qquad e_2 = (1, 0) \qquad e_3 = (1, 1) \tag{1}$$

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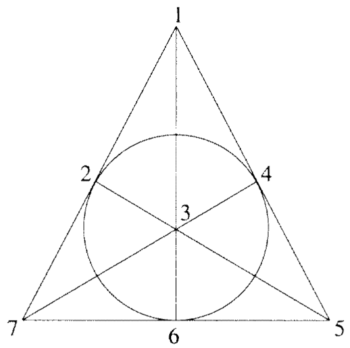

Figure 1. Seven-point plane.

with the multiplication rule,

$$e_i e_j = \varepsilon^2_{ijk} e_k \tag{2}$$

where ε_{ijk} is the structure constant of the $su(2)$ (quaternion) algebra. Using this structure constant, we obtain the three-dimensional Euler top equations with variables $(\omega_1(t), \omega_2(t), \omega_3(t))$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_i = \frac{1}{2}\varepsilon^2_{ijk}\omega_j\omega_k. \tag{3}$$

In the seven-dimensional ($n = 3$) case, we have seven points,

$$\begin{array}{llll} e_1 = (0,0,1) & e_2 = (0,1,0) & e_3 = (1,0,0) & e_4 = (1,1,1) \\ e_5 = (1,1,0) & e_6 = (1,0,1) & e_7 = (0,1,1) & \end{array} \tag{4}$$

with the relation

$$e_i e_j = c^2_{ijk} e_k \tag{5}$$

where c_{ijk} is equal to a realization of the totally antisymmetric structure constant appearing in the Cayley (octonion) algebra,

$$c_{127} = c_{631} = c_{541} = c_{532} = c_{246} = c_{734} = c_{567} = 1 \qquad \text{(others zero)}. \tag{6}$$

The relation (5) can be read off from the diagram in figure 1, the seven-point plane with seven points and seven lines; three points lie on each line and three lines pass through each point. Replacing ε_{ijk} in (3) by the constant c_{ijk}, we obtain the set of seven equations for a seven-dimensional top [1, 2],

$$\begin{array}{ll} \frac{\mathrm{d}}{\mathrm{d}t}\omega_1 = \omega_2\omega_7 + \omega_6\omega_3 + \omega_5\omega_4 & \frac{\mathrm{d}}{\mathrm{d}t}\omega_2 = \omega_7\omega_1 + \omega_5\omega_3 + \omega_4\omega_6 \\ \frac{\mathrm{d}}{\mathrm{d}t}\omega_3 = \omega_1\omega_6 + \omega_2\omega_5 + \omega_4\omega_7 & \frac{\mathrm{d}}{\mathrm{d}t}\omega_4 = \omega_1\omega_5 + \omega_6\omega_2 + \omega_7\omega_3 \\ \frac{\mathrm{d}}{\mathrm{d}t}\omega_5 = \omega_4\omega_1 + \omega_3\omega_2 + \omega_6\omega_7 & \frac{\mathrm{d}}{\mathrm{d}t}\omega_6 = \omega_3\omega_1 + \omega_2\omega_4 + \omega_7\omega_5 \\ \frac{\mathrm{d}}{\mathrm{d}t}\omega_7 = \omega_1\omega_2 + \omega_3\omega_4 + \omega_5\omega_6. & \end{array} \tag{7}$$

In a similar fashion to the above three-dimensional and seven-dimensional cases, we can obtain $2^n - 1$ equations for a $(2^n - 1)$-dimensional top. The structure of the higher-dimensional tops can be understood from the $(2^n - 1)$-point hyperplane diagram, which is an extension of the seven-point plane and consists of $2^n - 1$ points and $2^n - 1$ $(2^{n-1} - 1)$-point hyperplanes, where $2^{n-1} - 1$ points lie on each $(2^{n-1} - 1)$-point plane and $2^{n-1} - 1$ $(2^{n-1} - 1)$-point hyperplanes pass through each point.

For example, in the 15-dimensional ($n = 4$) case with an appropriate labelling of 15 points in Z_2P_3, we have a 15-point tetrahedral space containing the following 15 seven-point planes assigned by seven points in them,

$$
\begin{array}{lll}
(1, 2, 3, 4, 5, 6, 7) & (1, 2, 8, 11, 10, 9, 7) & (1, 3, 8, 13, 12, 9, 6) \\
(2, 3, 8, 14, 12, 10, 5) & (1, 2, 13, 14, 15, 12, 7) & (1, 3, 14, 11, 10, 15, 6) \\
(1, 4, 8, 14, 15, 9, 5) & (1, 4, 13, 11, 10, 12, 5) & (2, 3, 11, 13, 15, 9, 5) \\
(2, 4, 8, 13, 15, 10, 6) & (2, 4, 11, 14, 12, 9, 6) & (3, 4, 8, 11, 15, 12, 7) \\
(3, 4, 9, 10, 14, 13, 7) & (5, 6, 8, 11, 13, 14, 7) & (5, 6, 9, 10, 12, 15, 7)
\end{array}
\tag{8}
$$

where the pth element in each of the above 15 brackets is placed on the point p in figure 1. The form of the 15-dimensional top equations derived from the 15-point hyperplane is

$$
\begin{aligned}
\dot{\omega}_1 &= \omega_2\omega_7 + \omega_3\omega_6 + \omega_5\omega_4 + \omega_8\omega_9 + \omega_{10}\omega_{11} + \omega_{12}\omega_{13} + \omega_{14}\omega_{15} \\
\dot{\omega}_2 &= \omega_1\omega_7 + \omega_3\omega_5 + \omega_4\omega_6 + \omega_8\omega_{10} + \omega_{11}\omega_9 + \omega_{12}\omega_{14} + \omega_{15}\omega_{13} \\
\dot{\omega}_3 &= \omega_1\omega_6 + \omega_2\omega_5 + \omega_7\omega_4 + \omega_8\omega_{12} + \omega_9\omega_{13} + \omega_{10}\omega_{14} + \omega_{11}\omega_{15} \\
\dot{\omega}_4 &= \omega_5\omega_1 + \omega_2\omega_6 + \omega_7\omega_3 + \omega_8\omega_{15} + \omega_9\omega_{14} + \omega_{10}\omega_{13} + \omega_{11}\omega_{12} \\
\dot{\omega}_5 &= \omega_1\omega_4 + \omega_2\omega_3 + \omega_7\omega_6 + \omega_8\omega_{14} + \omega_9\omega_{15} + \omega_{10}\omega_{12} + \omega_{11}\omega_{13} \\
\dot{\omega}_6 &= \omega_1\omega_3 + \omega_2\omega_4 + \omega_7\omega_5 + \omega_8\omega_{13} + \omega_9\omega_{12} + \omega_{10}\omega_{15} + \omega_{11}\omega_{14} \\
\dot{\omega}_7 &= \omega_1\omega_2 + \omega_3\omega_4 + \omega_6\omega_5 + \omega_8\omega_{11} + \omega_9\omega_{10} + \omega_{12}\omega_{15} + \omega_{13}\omega_{14} \\
\dot{\omega}_8 &= \omega_1\omega_9 + \omega_2\omega_{10} + \omega_3\omega_{12} + \omega_4\omega_{15} + \omega_5\omega_{14} + \omega_6\omega_{13} + \omega_7\omega_{11} \\
\dot{\omega}_9 &= \omega_1\omega_8 + \omega_2\omega_{11} + \omega_3\omega_{13} + \omega_4\omega_{14} + \omega_5\omega_{15} + \omega_6\omega_{12} + \omega_7\omega_{10} \\
\dot{\omega}_{10} &= \omega_1\omega_{11} + \omega_2\omega_8 + \omega_3\omega_{14} + \omega_4\omega_{13} + \omega_5\omega_{12} + \omega_6\omega_{15} + \omega_7\omega_9 \\
\dot{\omega}_{11} &= \omega_1\omega_{10} + \omega_2\omega_9 + \omega_3\omega_{15} + \omega_4\omega_{12} + \omega_5\omega_{13} + \omega_6\omega_{14} + \omega_7\omega_8 \\
\dot{\omega}_{12} &= \omega_1\omega_{13} + \omega_2\omega_{14} + \omega_3\omega_8 + \omega_4\omega_{11} + \omega_5\omega_{10} + \omega_6\omega_9 + \omega_7\omega_{15} \\
\dot{\omega}_{13} &= \omega_1\omega_{12} + \omega_2\omega_{15} + \omega_3\omega_9 + \omega_4\omega_{10} + \omega_5\omega_{11} + \omega_6\omega_8 + \omega_7\omega_{14} \\
\dot{\omega}_{14} &= \omega_1\omega_{15} + \omega_2\omega_{12} + \omega_3\omega_{10} + \omega_4\omega_9 + \omega_5\omega_8 + \omega_6\omega_{11} + \omega_7\omega_{13} \\
\dot{\omega}_{15} &= \omega_1\omega_{14} + \omega_2\omega_{13} + \omega_3\omega_{11} + \omega_4\omega_8 + \omega_5\omega_9 + \omega_6\omega_{10} + \omega_7\omega_{12}.
\end{aligned}
\tag{9}
$$

2. General solution of the $(2^n - 1)$-dimensional top

2.1. Integrability

To show the integrability of the $(2^n - 1)$-dimensional top and its general solution, it is convenient to work with a set of $2^n - 1$ variables a_i, instead of the ω_i's. The rule to define the a_i's is to pick up $2^n - 1$ sets of 2^{n-1} ω_i's which do not lie on a $(2^{n-1} - 1)$-point subplane in the $(2^n - 1)$-point hyperplane and to assign a_i to the sum of all 2^{n-1} ω_i's in each of the $2^n - 1$ sets. For example, in the three-dimensional case,

$$a_1 = \omega_2 + \omega_3 \qquad a_2 = \omega_3 + \omega_1 \qquad a_3 = \omega_1 + \omega_2 \tag{10}$$

and in the seven-dimensional case,

$$
\begin{aligned}
&a_1 = \omega_3 + \omega_4 + \omega_5 + \omega_6 && a_2 = \omega_1 + \omega_2 + \omega_5 + \omega_6 \\
&a_3 = \omega_1 + \omega_3 + \omega_5 + \omega_7 && a_4 = \omega_2 + \omega_4 + \omega_5 + \omega_7 \\
&a_5 = \omega_2 + \omega_3 + \omega_6 + \omega_7 && a_6 = \omega_1 + \omega_4 + \omega_6 + \omega_7 \\
&a_7 = \omega_1 + \omega_2 + \omega_3 + \omega_4.
\end{aligned}
\tag{11}
$$

Similarly, 15 variables a_i in the 15-dimensional top can be easily read off from the 15-point hyperplane defined in (8).

Using the variables a_i, the $(2^n - 1)$-dimensional top equations are re-expressed as

$$\dot{a}_i = a_i(S - a_i) \qquad S = \frac{1}{2^{n-1}} \sum_{j=1}^{2^n-1} a_j \tag{12}$$

and the equations of motion for the difference of the a_i's are

$$(a_i \dot{-} a_k) = (a_i - a_k)(S - a_i - a_k). \tag{13}$$

We introduce the quantity W with the constants ρ_i and χ_{ij},

$$W = \sum_i \rho_i \ln a_i + \sum_{i<j} \chi_{ij} \ln (a_i - a_j). \tag{14}$$

The condition $\dot{W} = 0$ leads us to $(2^n - 1)(2^{n-1} - 1)$ conserved quantities N_{ij},

$$N_{ij} = T(a_i - a_j)/a_i a_j \qquad T = \left(\prod_{k=1}^{2^n-1} a_k \right)^{\frac{1}{2^{n-1}-1}}. \tag{15}$$

Although the N_{ij} are not independent, they are sufficient to construct a basis of $2^n - 2$ independent conserved quantities, thus guaranteeing the integrability of the $(2^n - 1)$-dimensional top. Specifically, all the N_{ij} can be expressed in terms of N_{1j} $(j = 2, \ldots, 2^n - 1)$ through the relation

$$N_{ij} = N_{1j} - N_{1i} \tag{16}$$

which means that any conserved quantities in the system can be constructed from these $2^n - 2$ quantities N_{1j}. In particular it is possible to define $2^n - 1$ polynomial conserved quantities γ_i from N_{ij} as

$$\gamma_i = N_{j_1 k_1} N_{j_2 k_2} \ldots N_{j_{2^{n-1}-1} k_{2^{n-1}-1}} = a_i (a_{j_1} - a_{k_1})(a_{j_2} - a_{k_2}) \ldots (a_{j_{2^{n-1}-1}} - a_{k_{2^{n-1}-1}}) \tag{17}$$

where (j_p, k_p), $(j_p < k_p,\ p = 1, \ldots, 2^{n-1} - 1)$ lie on the respective $2^{n-1} - 1$ lines through the point i. The polynomials γ_i are of order 2^{n-1}. There is, of course, one functional relationship connecting these $2^n - 1$ expressions.

Summing over the index i of N_{ij} in (15), we see that all a_j are expressed in terms of two variables T and U, with the constants $M_j = \sum_{i=1}^{2^n-1} N_{ij}/(2^n - 1)$,

$$a_j^{-1} = M_j T^{-1} + U \qquad U = \frac{1}{2^n - 1} \sum_{i=1}^{2^n-1} a_i^{-1}. \tag{18}$$

Note that the variables T and U are symmetric under any permutation of a_i's. Substituting the expression of a_i's into the definition of T in (15), we have the following relation between T and U,

$$T^{2^{n-1}} = \prod_{j=1}^{2^n-1} (TU + M_j). \tag{19}$$

From (18) and (19), we see that all variables are expressible in terms of one variable, which demonstrates that the system of the $(2^n - 1)$-dimensional top is integrable. The explicit expression for the quadrature whose evaluation solves the top is given in section 2.2.

2.2. General solution

The time derivatives of T and U are derived from the equations of motion (12),

$$\dot{T} = TS \qquad \dot{U} = -US + 1. \tag{20}$$

We introduce a variable $R(t) = T(t)\,U(t)$, whose time derivative is given as

$$\dot{R} = T. \tag{21}$$

Substituting $R = T\,U$ into (19) and (18), we have

$$T = \left(\prod_{j=1}^{2^n-1}(R+M_j)\right)^{\frac{1}{2^{n-1}}} \tag{22}$$

and

$$a_j = \frac{T}{(R+M_j)} = \frac{(\prod_{k=1}^{2^n-1}(R+M_k))^{\frac{1}{2^{n-1}}}}{(R+M_j)}. \tag{23}$$

Using (21) and (22), we obtain a first-order equation for $R(t)$,

$$\dot{R} = \left(\prod_{j=1}^{2^n-1}(R+M_j)\right)^{\frac{1}{2^{n-1}}} \tag{24}$$

which is non-hyperelliptic except for the three-dimensional ($n = 2$) case. The integral associated with this equation can be shown to correspond to a Riemann surface with genus $g = (2^{n-1} - 1)^2$; the order $1/2^{n-1}$ in the RHS of (24) means that we need 2^{n-1} complex surfaces, each of which has 2^{n-1} cuts since the order of R is $2^n - 1$ inside the bracket of the RHS.

3. Further generalizations

It would be natural to expect that the examples of this note could be further generalized to the discussion of evolution equations for $\frac{k^n-1}{k-1}$ variables, corresponding to tops based upon the space $Z_k P_{n-1}$. Despite many efforts, we have as yet been unable to demonstrate a set of integrable equations for integral $k > 2$ except for the case where $n = 2$ and there are $k+1$ points lying on a line. Then one possibility for a set of integrable evolution equations is [6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_i = \prod_{j\neq i}\omega_j \qquad (i = 1, \ldots, k+1). \tag{25}$$

These equations are reduced to a hyperelliptic differential equation for a $g = k-1$ Riemann surface.

An alternative approach to further generalization would be based upon Lie algebras. It was pointed out to us by Jan Govaerts, and elaborated in further discussions with Ryu Sasaki that the RHS of our equations for the $(2^n - 1)$-dimensional top can be interpreted as a product rule among weight vectors in the $B_n = SO(2n+1)$ Lie algebra; the product of two spinor representations with the dimension 2^n gives $2^n - 1$ positive weight vectors in the first quadrant in the space R^n, $m_1\boldsymbol{k}_1 + \cdots + m_n\boldsymbol{k}_n$ ($m_i = 0, 1$), where $\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\}$ is an orthonormal basis in R^n. These $2^n - 1$ vectors just correspond to the $2^n - 1$ points in the space $Z_2 P_{n-1}$. It is intriguing to explore a generalization of this result to the other classical Lie algebras and their representations. Work along these lines is under consideration.

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